

## Abstract

By using the  $q$ -analogue of van der Corput's method we study the divisor function in an arithmetic progression to modulus  $q$ . We show that the expected asymptotic formula holds for a larger range of  $q$  than was previously known, provided that  $q$  has a certain factorisation.

## 1 Introduction

Given an arithmetic function  $f(n)$  it is natural to consider the sum

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} f(n).$$

For many functions  $f$  we might hope to show that when  $(a, q) = 1$  this is asymptotic to

$$\frac{1}{\varphi(q)} \sum_{\substack{n \leq x \\ (n, q) = 1}} f(n).$$

In applications it is often essential that we establish such a result uniformly in  $q \leq x^\theta$  with  $\theta$  as large as possible.

In this paper we will consider the divisor function  $\tau(n)$ , which counts the number of positive divisors of  $n$ . We therefore let

$$D(x, q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \tau(n)$$

and

$$D(x, q) = \frac{1}{\varphi(q)} \sum_{\substack{n \leq x \\ (n, q) = 1}} \tau(n).$$

We then wish to estimate  $E(x, q, a) = D(x, q, a) - D(x, q)$ . We hope to show that for some  $\delta > 0$  we have the bound

$$E(x, q, a) \ll \frac{x^{1-\delta}}{q}. \quad (1)$$

If  $q \leq x^{2/3-\eta}$  for some  $\eta > 0$  then (1) holds with a  $\delta$  depending on  $\eta$ . This was proved independently in unpublished work of Hooley, Linnik and Selberg, it is a consequence of the Weil bound for Kloosterman sums. For larger  $q$ , no nontrivial bound is known for individual  $E(x, q, a)$  but there are various results on average. For example Fouvry [3, Corollaire 5] showed that for any  $\eta, A > 0$  and any  $a \in \mathbb{Z}$  we have

$$\sum_{\substack{x^{2/3+\eta} \leq q \leq x^{1-\eta} \\ (q, a) = 1}} |E(x, q, a)| \ll_{A, a, \eta} x(\log x)^{-A}.$$

An average over moduli  $x^{2/3-\eta} \leq q \leq x^{2/3+\eta}$  was considered by Fouvry and Iwaniec in [5]. Their approach requires them to work only with moduli  $q$  which have a squarefree factor  $r$  of a certain size. Specifically, they show that if  $r$  is squarefree with  $r \leq x^{3/8}$  and  $(r, a) = 1$  then for any  $\eta > 0$  we have

$$\sum_{\substack{rs^2 \leq x^{1-6\eta} \\ (s, ar)=1}} |E(x, rs, a)| \ll_{\eta} r^{-1} x^{1-\eta}.$$

Observe that to handle moduli  $q = rs$  of size  $x^{2/3}$  with this result it is necessary that  $r \geq x^{1/3+6\eta}$ . Further results are possible if we exploit averaging over the residue class  $a \pmod{q}$ . See for example Banks, Heath-Brown and Shparlinski [1] and Blomer [2].

We will show that (1) holds for an individual  $E(x, q, a)$  for  $q$  almost as large as  $x^{\frac{55}{82}}$  provided that  $q$  factorises in a certain way. This will follow by optimising the sizes of the parameters in the following result.

**Theorem 1.1.** *Suppose that  $q = q_0 q_1 q_2 q_3$  is squarefree and  $(a, q) = 1$ . For any  $x \geq q$ ,  $\delta \in (0, \frac{1}{12})$  and any  $\epsilon > 0$  we have*

$$E(x, q, a) \ll_{\epsilon} q^{-1} x^{1-\delta+\epsilon} + x^{2\delta+\epsilon} \left( \sum_{j=1}^3 x^{2^{-j}-1} q^{1/2-2^{-j}} q_{4-j}^{2^{-j}} + x^{1/16} q^{3/8} q_0^{1/16} + q^{1/2} q_0^{-1/16} \right).$$

It is not immediately clear when the estimate in this theorem is nontrivial. We therefore prove the following, in which we exploit the fact that if  $q$  is sufficiently smooth then we can find a suitable factorisation for which our bound is close to optimal.

**Theorem 1.2.** *Suppose  $\varpi, \eta > 0$  satisfy*

$$246\varpi + 18\eta < 1.$$

*There exists a  $\delta > 0$ , depending on  $\varpi$  and  $\eta$ , such that for any  $x^{\eta}$ -smooth, squarefree  $q \leq x^{2/3+\varpi}$  and any  $(a, q) = 1$  we have*

$$E(x, q, a) \ll_{\varpi, \eta} q^{-1} x^{1-\delta}.$$

Observe that for any  $\varpi < \frac{1}{246}$  this theorem shows that there is an  $\eta > 0$  for which the conclusion holds. This means that we get a bound for sufficiently smooth  $q$  which are almost as large as  $x^{\frac{55}{82}}$ . The smoothness assumption is not necessary, it is simply a convenient way of guaranteeing that suitably sized factors exist. For example, given a squarefree  $q \sim x^{2/3}$ , Theorem 1.1 gives a nontrivial estimate provided that, for some  $\eta > 0$ , we have  $q = q_0 q_1 q_2 q_3$  with

$$x^{\eta} \leq q_0 \leq x^{1/3-\eta}$$

and

$$q_j \leq x^{1/6-\eta} \text{ for } 1 \leq j \leq 3.$$

Writing

$$e_q(x) = e^{\frac{2\pi i x}{q}},$$

the proof of Theorem 1.1 depends on estimates for short Kloosterman sums

$$\sum_{\substack{n \in I \\ (n, q) = 1}} e_q(b\bar{n}),$$

where  $b$  is an integer and  $I$  is an interval of length  $O(\sqrt{x})$ . If  $(b, q) = 1$  then the Weil bound gives an estimate of  $O_\epsilon(q^{1/2+\epsilon})$  for such a sum. For the sizes of  $x$  and  $q$  in which we are interested this is a significant saving over the trivial bound of  $\sqrt{x}$ . In particular it is enough to estimate  $E(x, q, a)$  if  $q \leq x^{2/3-\eta}$ . For larger  $q$  we must improve upon the Weil estimate. This is achieved for special  $q$  by means of the following result.

**Theorem 1.3.** *Let  $q = q_0 q_1 \dots q_l$  be squarefree. Suppose that  $(a, q) = 1$  and that  $I$  is an interval of length at most  $N \leq q$ . Let*

$$S = \sum_{\substack{n \in I \\ (n, q) = 1}} e_q(a\bar{n}).$$

For any  $\epsilon > 0$  we have

$$S \ll_{\epsilon, l} q^\epsilon \left( \sum_{j=1}^l N^{2^{-j}} q^{1/2-2^{-j}} q_{l-j+1}^{2^{-j}} + N^{2^{-l}} q^{1/2-2^{-l}} q_0^{1/2^{l+1}} + q^{1/2} q_0^{-1/2^{l+1}} \right).$$

This theorem is very similar to that of Heath-Brown [6, Theorem 2]. His result can be applied to our sum  $S$  to obtain the bound

$$S \ll_{\epsilon, l} q^\epsilon \left( \sum_{j=1}^l N^{1-2^{-j}} q_{l-j+1}^{2^{-j}} + N^{1-2^{-l}} q_0^{1/2^{l+1}} + N q_0^{-1/2^{l+1}} \right).$$

When the sizes of the factors  $q_j$  are chosen optimally this result of Heath-Brown is nontrivial provided that  $N$  is approximately  $q^{\frac{1}{l+1}}$ . In contrast, our bound is most useful when  $N \approx q^{1-\frac{1}{l+1}}$  in which case it can improve on the Weil bound.

As in Heath-Brown's work our proof of Theorem 1.3 uses the  $q$ -analogue of van der Corput's method. We begin by completing the sum  $S$  and then apply the differencing process  $l$ -times, whereas Heath-Brown applied differencing directly to  $S$ . The result is a sum of products of  $2^l$  Kloosterman sums which we estimate by another completion followed by the application of a bound for complete exponential sums due to Fouvry, Ganguly, Kowalski and Michel [4]. In other words, our result is a  $q$ -analogue of the  $BA^l B$  van der Corput estimate whereas Heath-Brown's is analogous to  $A^l B$ . A  $q$ -analogue of  $BA^2 B$  was used by Heath-Brown in [7] but the exponential sums in that work are not Kloosterman sums.

The assumption that  $q$  is squarefree is important for two reasons. Firstly, it guarantees that the factors  $q_j$  are coprime in pairs, thereby avoiding many unpleasant technicalities. Secondly, it means that we need only consider complete exponential sums to prime moduli. To handle  $q$  which are not squarefree Lemma 4.4 would have to be generalised to prime-power moduli.

Throughout this work we use the notation  $x \sim y$  for the inequality  $y \leq x < 2y$ . We adopt the standard convention that  $\epsilon$  denotes a sufficiently small positive quantity whose value may differ at each occurrence.

## Acknowledgements

This work was completed as part of my DPhil, for which I was funded by EPSRC grant EP/P505666/1. I am very grateful to the EPSRC for funding me and to my supervisor, Roger Heath-Brown, for all his valuable help and advice. I would also like to thank Emmanuel Kowalski for his assistance with the complete exponential sums arising in this work.

## 2 Proof of Theorem 1.1

In this section we will show that Theorem 1.1 follows from Theorem 1.3. Recall that we wish to estimate

$$E(x, q, a) = \sum_{\substack{uv \leq x \\ uv \equiv a \pmod{q}}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{uv \leq x \\ (uv, q) = 1}} 1.$$

By a dyadic subdivision it is enough to consider each of the  $O((\log x)^2)$  sums of the form

$$E_1(U, V, q, a) = \sum_{\substack{u \sim U, v \sim V \\ uv \leq x, uv \equiv a \pmod{q}}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{u \sim U, v \sim V \\ uv \leq x, (uv, q) = 1}} 1 = D_1(U, V, q, a) - D_1(U, V, q),$$

say. We must bound  $E_1(U, V, q, a)$  for all  $U, V \geq 1$  for which  $UV \leq x$ . However, by symmetry we can assume that  $U \leq \sqrt{x}$ .

We will use a short interval decomposition to remove the constraint  $uv \leq x$  from  $D_1(U, V, q, a)$  and  $D_1(U, V, q)$ . Specifically we divide the range  $u \sim U$  into  $O(x^\delta)$  intervals of length  $Ux^{-\delta}$  and the range  $v \sim V$  into  $O(x^\delta)$  intervals of length  $Vx^{-\delta}$ . We will denote the resulting intervals by

$$I_1(U_1) = [U_1, U_1 + Ux^{-\delta})$$

and

$$I_2(V_1) = [V_1, V_1 + Vx^{-\delta}).$$

We only need consider the case that  $U_1V_1 \leq x$ . Dropping the constraint  $uv \leq x$  has the effect of including in the above sums points  $(u, v) \in I_1(U_1) \times I_2(V_1)$  with

$$x < uv \leq (U_1 + Ux^{-\delta})(V_1 + Vx^{-\delta}) \leq x + O(x^{1-\delta}).$$

It follows that the errors introduced by removing the constraint are bounded by

$$\sum_{\substack{x < n \leq x + O(x^{1-\delta}) \\ n \equiv a \pmod{q}}} \tau(n) \ll_{\epsilon} q^{-1} x^{1-\delta+\epsilon}$$

and

$$\frac{1}{\varphi(q)} \sum_{\substack{x < n \leq x + O(x^{1-\delta}) \\ (n, q) = 1}} \tau(n) \ll_{\epsilon} q^{-1} x^{1-\delta+\epsilon}.$$

We conclude that it is enough to bound  $O(x^{2\delta}(\log x)^2)$  sums of the form

$$E_2(U_1, V_1, q, a) = D_2(U_1, V_1, q, a) - D_2(U_1, V_1, q)$$

where

$$D_2(U_1, V_1, q, a) = \#\{u \in I_1(U_1), v \in I_2(V_1) : uv \equiv a \pmod{q}\}$$

and

$$D_2(U_1, V_1, q) = \frac{1}{\varphi(q)} \#\{u \in I_1(U_1), v \in I_2(V_1) : (uv, q) = 1\}.$$

Specifically we have

$$E(x, q, a) \ll_{\epsilon} q^{-1} x^{1-\delta+\epsilon} + x^{2\delta+\epsilon} \max_{U_1, V_1} |E_2(U_1, V_1, q, a)|.$$

We now write

$$\begin{aligned} D_2(U_1, V_1, q, a) &= \sum_{\substack{u \in I_1(U_1), v \in I_2(V_1) \\ uv \equiv a \pmod{q}}} 1 \\ &= \sum_{\substack{u \in I_1(U_1) \\ (u, q) = 1}} \sum_{\substack{v \in I_2(V_1) \\ v \equiv a\bar{u} \pmod{q}}} 1 \\ &= \frac{1}{q} \sum_{\substack{u \in I_1(U_1) \\ (u, q) = 1}} \sum_{v \in I_2(V_1)} \sum_{k=1}^q e_q(k(a\bar{u} - v)) \\ &= \frac{1}{q} \sum_{k=1}^q \left( \sum_{\substack{u \in I_1(U_1) \\ (u, q) = 1}} e_q(ak\bar{u}) \right) \left( \sum_{v \in I_2(V_1)} e_q(-kv) \right). \end{aligned}$$

The  $k = q$  terms in this are

$$\frac{1}{q} \#\{u \in I_1(U_1) : (u, q) = 1\} \#I_2(V_1).$$

On the other hand

$$\begin{aligned}
D_2(U_1, V_1, q) &= \frac{1}{\varphi(q)} \sum_{\substack{u \in I_1(U_1) \\ (u, q) = 1}} \sum_{\substack{v \in I_2(V_1) \\ (v, q) = 1}} 1 \\
&= \frac{1}{\varphi(q)} \sum_{\substack{u \in I_1(U_1) \\ (u, q) = 1}} \left( \frac{\varphi(q)}{q} \#I_2(V_1) + O_\epsilon(q^\epsilon) \right) \\
&= \frac{1}{q} \# \{u \in I_1(U_1) : (u, q) = 1\} \#I_2(V_1) + O_\epsilon(q^{-1+\epsilon} x^{1/2}),
\end{aligned}$$

where we have used our assumption that  $U \leq \sqrt{x}$ . Since  $\delta < \frac{1}{6}$  and  $q \leq x$  we have

$$x^{2\delta} \cdot q^{-1+\epsilon} x^{1/2} < q^{-1} x^{1-\delta+\epsilon}$$

so we conclude that the  $k = q$  terms correspond to  $D_2(U_1, V_1, q)$  with a sufficiently small error.

It remains to bound

$$\frac{1}{q} \sum_{k=1}^{q-1} \left| \sum_{\substack{u \in I_1(U_1) \\ (u, q) = 1}} e_q(ak\bar{u}) \right| \left| \sum_{v \in I_2(V_1)} e_q(-kv) \right|.$$

We write this as

$$\begin{aligned}
&\frac{1}{q} \sum_{d|q} \sum_{\substack{k=1 \\ (k, q) = d}}^{q-1} \left| \sum_{\substack{u \in I_1(U_1) \\ (u, q) = 1}} e_q(ak\bar{u}) \right| \left| \sum_{v \in I_2(V_1)} e_q(-kv) \right| \\
&= \frac{1}{q} \sum_{\substack{d|q \\ d < q}} \sum_{k \pmod{q/d}}^* \left| \sum_{\substack{u \in I_1(U_1) \\ (u, q) = 1}} e_{q/d}(ak\bar{u}) \right| \left| \sum_{v \in I_2(V_1)} e_{q/d}(-kv) \right|.
\end{aligned}$$

However, since  $q$  is squarefree we have

$$\begin{aligned}
\sum_{\substack{u \in I_1(U_1) \\ (u, q) = 1}} e_{q/d}(ak\bar{u}) &= \sum_{\substack{u \in I_1(U_1) \\ (u, q/d) = 1}} e_{q/d}(ak\bar{u}) \sum_{e|(d, u)} \mu(e) \\
&= \sum_{e|d} \mu(e) \sum_{\substack{u \in I_1(U_1)/e \\ (u, q/d) = 1}} e_{q/d}(ak\bar{e}u).
\end{aligned}$$

Our sum is therefore bounded by

$$\frac{1}{q} \sum_{\substack{d|q \\ d < q}} \sum_{k \pmod{q/d}}^* \left| \sum_{v \in I_2(V_1)} e_{q/d}(-kv) \right| \sum_{e|d} \left| \sum_{\substack{u \in I_1(U)/e \\ (u, q/d)=1}} e_{q/d}(ak\overline{eu}) \right|.$$

We have the standard estimate

$$\sum_{v \in I_2(V_1)} e_{q/d}(-kv) \ll \min \left( Vx^{-\delta}, \frac{1}{\|dk/q\|} \right)$$

so that this is at most

$$\begin{aligned} & \frac{1}{q} \sum_{\substack{d|q \\ d < q}} \sum_{e|d} \max_{(b, q/d)=1} \left| \sum_{\substack{u \in I_1(U)/e \\ (u, q/d)=1}} e_{q/d}(b\overline{u}) \right| \sum_{k \pmod{q/d}}^* \frac{1}{\|dk/q\|} \\ & \ll_{\epsilon} q^{\epsilon} \sum_{\substack{d|q \\ d < q}} \frac{1}{d} \sum_{e|d} \max_{(b, q/d)=1} \left| \sum_{\substack{u \in I_1(U)/e \\ (u, q/d)=1}} e_{q/d}(b\overline{u}) \right|. \end{aligned}$$

To estimate the contribution to this from  $d \geq qx^{-2/3+2\delta}$  we apply the Weil bound which gives

$$\max_{(b, q/d)=1} \left| \sum_{\substack{u \in I_1(U_1)/e \\ (u, q/d)=1}} e_{q/d}(b\overline{u}) \right| \ll_{\epsilon} \frac{Ux^{-\delta}d}{qe} + (q/d)^{1/2+\epsilon}.$$

The contribution to our sum from such  $d$  is therefore bounded by

$$q^{\epsilon} \sum_{\substack{d|q \\ qx^{-2/3+2\delta} \leq d < q}} \left( \frac{Ux^{-\delta}}{q} + q^{1/2}/d^{3/2} \right) \ll_{\epsilon} q^{\epsilon} (Ux^{-\delta}q^{-1} + q^{-1}x^{1-3\delta}).$$

The contribution of these  $d$  to  $E(x, q, a)$  is therefore  $O_{\epsilon}(q^{-1}x^{1-\delta+\epsilon})$ . If  $q < x^{2/3-2\delta}$  then this analysis covers all values of  $d$  and therefore completes the proof.

If  $q \geq x^{2/3-2\delta}$  and  $d < qx^{-2/3+2\delta}$  we apply Theorem 1.3 with  $l = 3$  and the factorisation

$$\frac{q}{d} = \prod_{j=0}^3 \frac{q_j}{(q_j, d)},$$

which holds since  $q$  is squarefree. We have

$$\frac{q}{d} \geq x^{2/3-2\delta} \geq \sqrt{x},$$

since  $\delta < \frac{1}{12}$ . We may therefore deduce that if  $(b, q/d) = 1$  then

$$\frac{1}{d} \sum_{\substack{u \in I_1(U)/e \\ (u, q/d)=1}} e_{q/d}(b\bar{u}) \ll_{\epsilon} q^{\epsilon} \left( \sum_{j=1}^3 x^{2^{-j}-1} q^{1/2-2^{-j}} q_{4-j}^{2^{-j}} + x^{1/16} q^{3/8} q_0^{1/16} + q^{1/2} q_0^{-1/16} \right).$$

It follows that we have

$$\begin{aligned} & q^{\epsilon} \sum_{\substack{d|q \\ d < qx^{-2/3+2\delta}}} \frac{1}{d} \sum_{e|d} \max_{(b, q/d)=1} \left| \sum_{\substack{u \in I_1(U)/e \\ (u, q/d)=1}} e_{q/d}(b\bar{u}) \right| \\ & \ll_{\epsilon} q^{\epsilon} \left( \sum_{j=1}^3 x^{2^{-j}-1} q^{1/2-2^{-j}} q_{4-j}^{2^{-j}} + x^{1/16} q^{3/8} q_0^{1/16} + q^{1/2} q_0^{-1/16} \right). \end{aligned}$$

We conclude that the contribution of this to  $E(x, q, a)$  is majorised by

$$x^{2\delta+\epsilon} \left( \sum_{j=1}^3 x^{2^{-j}-1} q^{1/2-2^{-j}} q_{4-j}^{2^{-j}} + x^{1/16} q^{3/8} q_0^{1/16} + q^{1/2} q_0^{-1/16} \right).$$

This completes the proof of Theorem 1.1.

### 3 Proof of Theorem 1.2

Suppose  $\varpi, \eta, q$  and  $a$  are as in Theorem 1.2. Let  $\delta > 0$  be a parameter which we will eventually choose to be very small. We may suppose that  $q \geq x^{2/3-2\delta}$  since the result is known for smaller  $q$ . Applying Theorem 1.1 we deduce that for any  $\epsilon > 0$  we have

$$E(x, q, a) \ll_{\epsilon} q^{-1} x^{1-\delta+\epsilon} + x^{2\delta+\epsilon} \left( \sum_{j=1}^3 x^{2^{-j}-1} q^{1/2-2^{-j}} q_{4-j}^{2^{-j}} + x^{1/16} q^{3/8} q_0^{1/16} + q^{1/2} q_0^{-1/16} \right).$$

The first term in this is sufficiently small. We optimise the remaining terms by working with a factorisation for which  $q_j \approx Q_j$  with

$$Q_0 = q^{-2/15} x^{1/3},$$

$$Q_1 = q^{-1/15} x^{1/6},$$

$$Q_2 = q^{7/15} x^{-1/6}$$

and

$$Q_3 = q^{11/15} x^{-1/3}.$$



Observe that  $Q_0Q_1Q_2Q_3 = q$  and that for all sufficiently small  $\delta$  we have  $Q_j > x^{1/18} > x^\eta$  for all  $j$ . Since  $q$  is  $x^\eta$ -smooth we may find a factorisation  $q = q_0q_1q_2q_3$  with

$$q_1 \in [Q_1x^{-\eta/5}, Q_1x^{4\eta/5}],$$

$$q_2 \in [Q_2x^{-3\eta/5}, Q_2x^{2\eta/5}],$$

$$q_3 \in [Q_3x^{-4\eta/5}, Q_3x^{\eta/5}]$$

so that

$$q_0 \in [Q_0x^{-7\eta/5}, Q_0x^{8\eta/5}].$$

This gives

$$E(x, q, a) \ll_\epsilon q^{-1}x^{1-\delta+\epsilon} + x^{2\delta+\epsilon} \left( x^{1/12+\eta/10} q^{11/30} + x^{-1/48+7\eta/80} q^{61/120} \right).$$

Finally, recalling that  $q \leq x^{2/3+\varpi}$  we get

$$\begin{aligned} E(x, q, a) &\ll_\epsilon q^{-1}x^{1-\delta+\epsilon} + q^{-1}x^{2\delta+\epsilon} \left( x^{1/12+\eta/10} q^{41/30} + x^{-1/48+7\eta/80} q^{181/120} \right) \\ &\ll_\epsilon q^{-1}x^{1-\delta+\epsilon} + q^{-1}x^{2\delta+\epsilon} \left( x^{\frac{179+18\eta+246\varpi}{180}} + x^{\frac{709+63\eta+1086\varpi}{720}} \right). \end{aligned}$$

We know that

$$246\varpi + 18\eta < 1.$$

In particular  $\varpi < \frac{1}{246}$  and  $\eta < \frac{1}{18}$  so

$$63\eta + 1086\varpi < \frac{649}{82} < 8.$$

Theorem 1.2 therefore follows on taking  $\delta$  and  $\epsilon$  sufficiently small in terms of  $\varpi$  and  $\eta$ .

## 4 Proof of Theorem 1.3

Suppose that for some  $1 \leq j \leq l$  we have  $[q/N] < q_{l-j+1}$ . Then

$$q^\epsilon N^{2^{-j}} q^{1/2-2^{-j}} q_{l-j+1}^{2^{-j}} \gg q^{1/2+\epsilon}.$$

Our result therefore follows from the Weil bound. We may therefore assume, for the remainder of the paper, that  $[q/N] \geq q_j$  for all  $1 \leq j \leq l$ .

## 4.1 Completion of $S$

Let  $f(k)$  be the Fourier transform of the interval  $I$ :

$$f(k) = \sum_{n \in I} e_q(-nk).$$

We have

$$S = \frac{1}{q} \sum_{k \pmod{q}} f(k) S(a, k, q)$$

where  $S(a, k, q)$  is the Kloosterman sum given by

$$S(a, k, q) = \sum_{n \pmod{q}}^* e_q(a\bar{n} + kn).$$

Since  $f(0) \ll N$  and  $S(a, 0, q) = \mu(q) \ll 1$  we get

$$S \ll \frac{N}{q} + \frac{1}{q} \left| \sum_{k \not\equiv 0 \pmod{q}} f(k) S(a, k, q) \right|.$$

The term  $N/q$  is clearly small enough.

We may assume that  $I \subseteq [M, M + N]$  for some integer  $M$ . We then write

$$f(k) = e_q(-kM) \sum_{\substack{n < N \\ n+M \in I}} e_q(-kn) = e_q(-kM) g(k),$$

say. Thus

$$S \ll \frac{N}{q} + \frac{1}{q} \left| \sum_{k \not\equiv 0 \pmod{q}} g(k) e_q(-kM) S(a, k, q) \right|.$$

We will consider the contribution to this bound from  $0 < k \leq q/2$ . One can use a completely analogous treatment for the range  $-q/2 < k < 0$ .

We wish to remove the weight  $g(k)$ . We have the standard estimate

$$g(k) \ll \min \left( N, \frac{1}{\|k/q\|} \right) = \min \left( N, \frac{q}{k} \right).$$

In addition

$$g'(k) = -2\pi i \sum_{\substack{n < N \\ n+M \in I}} \frac{n}{q} e_q(-kn) \ll \frac{N}{q} \min \left( N, \frac{q}{k} \right).$$

We will split the sum over  $k$  into intervals on which we may remove  $g(k)$  by partial summation. Specifically, let  $K = \lfloor q/N \rfloor$  and

$$S(r) = \max_{0 \leq L \leq K} \left| \sum_{(r-1)K < k \leq (r-1)K+L} e_q(-Mk) S(a, k, q) \right| \quad (r = 1, 2, 3, \dots).$$

Summing by parts we get, for any  $K' \leq K$  that

$$\sum_{(r-1)K \leq k \leq (r-1)K+K'} g(k) e_q(-Mk) S(a, k, q) \ll S(r) \min \left( N, \frac{q}{(r-1)K} \right) \ll \frac{N}{r} S(r).$$

It is therefore sufficient to estimate

$$\frac{N}{q} \sum_{r \leq N} \frac{S(r)}{r}$$

which we accomplish by bounding each  $S(r)$  individually. We will prove the following, which easily implies Theorem 1.3.

**Lemma 4.1.** *Under the hypotheses of Theorem 1.3 and with  $K, S(r)$  as above we have*

$$S(r) \ll_{\epsilon, l} q^{1/2+\epsilon} \left( \sum_{j=1}^l K^{1-2^{-j}} q_{l-j+1}^{2^{-j}} + K^{1-2^{-l}} q_0^{1/2^{l+1}} + K q_0^{-1/2^{l+1}} \right).$$

## 4.2 Differencing the Sum $S(r)$

In the remainder of the paper we will frequently use without comment the fact that, since  $q$  is squarefree, any pair of integers  $q', q''$  with  $q'q''|q$  must be coprime. We now apply a  $q$ -analogue of the van der Corput  $A$ -process. Let  $J$  be an interval whose length is bounded above by  $K$ . Suppose  $(a, q) = 1$  and  $s_1, \dots, s_j$  are integers, for some  $j \geq 1$ . We consider the more general sum

$$T = \sum_{k \in J} e_q(-Mk) \prod_{i=1}^j S(a, k + s_i, q)$$

in which the value of  $M$  may differ from that in  $S(r)$ . The sums  $S(r)$  correspond to the case  $j = 1$  and  $s_1 = 0$  of this. The following lemma describes a single van der Corput differencing step applied to the sum  $T$ . Note that the factors  $q_0, q_1$  occurring need not correspond to those in Theorem 1.3.

**Lemma 4.2.** *Suppose  $q = q_0 q_1$  with  $q_1 \leq K$ . We have*

$$T^2 \ll_{\epsilon, j} q^\epsilon q_1^{j+1} \left( K q_0^j + \sum_{0 < |h| \leq K/q_1} \left| \sum_{k \in J(h)} \prod_{i=1}^j S(a', k + s_i, q_0) S(a', k + q_1 h + s_i, q_0) \right| \right)$$

where  $J(h)$  is an interval of length at most  $K$  which depends on  $h$ , and where  $a' = a(\bar{q}_1)^2$ .

*Proof.* We let  $H = \lceil K/q_1 \rceil \geq 1$  and

$$a_k = \begin{cases} e_q(-Mk) \prod_{i=1}^j S(a, k + s_i, q) & k \in J \\ 0 & k \notin J. \end{cases}$$

Since  $H \geq 1$  we have

$$\begin{aligned} T &= \sum_k a_k \\ &= \frac{1}{H} \sum_{h=1}^H \sum_k a_{k+q_1 h} \\ &= \frac{1}{H} \sum_k \sum_{h=1}^H a_{k+q_1 h}. \end{aligned}$$

If  $k + q_1 h \in J$  then

$$\begin{aligned} a_{k+q_1 h} &= e_q(-M(k + q_1 h)) \prod_{i=1}^j S(a, k + q_1 h + s_i, q) \\ &= e_q(-Mk) e_q(-Mq_1 h) \prod_{i=1}^j S(a\bar{q}_1, (k + q_1 h + s_i)\bar{q}_1, q_0) S(a\bar{q}_0, (k + s_i)\bar{q}_0, q_1). \end{aligned}$$

Since  $q_1 H \leq K$  the sum over  $k$  is supported on an interval of length bounded by  $O(K)$ . By the Weil bound we get

$$S(a\bar{q}_0, (k + s_i)\bar{q}_0, q_1) \ll_{\epsilon} q_1^{1/2+\epsilon}.$$

Therefore, applying Cauchy's inequality, we obtain

$$H^2 T^2 \ll_{\epsilon, j} K q_1^{j+\epsilon} \sum_k \left| \sum_{\substack{h=1 \\ k+q_1 h \in J}}^H e_q(-Mq_1 h) \prod_{i=1}^j S(a\bar{q}_1, (k + q_1 h + s_i)\bar{q}_1, q_0) \right|^2.$$

Letting  $a' = a(\bar{q}_1)^2$ , as in the statement of the lemma, we have  $(a', q) = 1$  and

$$H^2 T^2 \ll_{\epsilon, j} K q_1^{j+\epsilon} \sum_k \left| \sum_{\substack{h=1 \\ k+q_1 h \in J}}^H e_q(-Mq_1 h) \prod_{i=1}^j S(a', k + q_1 h + s_i, q_0) \right|^2.$$

Expanding the square and reordering we deduce that

$$\begin{aligned}
H^2 T^2 &\ll_{\epsilon, j} K q_1^{j+\epsilon} \sum_{h_1, h_2=1}^H \left| \sum_{\substack{k \\ k+q_1 h_1, k+q_1 h_2 \in J}} \prod_{i=1}^j S(a', k+q_1 h_1+s_i, q_0) S(a', k+q_1 h_2+s_i, q_0) \right| \\
&= K q_1^{j+\epsilon} \sum_{h_1, h_2=1}^H \left| \sum_{\substack{k \in J \\ k+q_1(h_2-h_1) \in J}} \prod_{i=1}^j S(a', k+s_i, q_0) S(a', k+q_1(h_2-h_1)+s_i, q_0) \right| \\
&\leq K H q_1^{j+\epsilon} \sum_{|h| \leq H} \left| \sum_{\substack{k \in J \\ k+q_1 h \in J}} \prod_{i=1}^j S(a', k+s_i, q_0) S(a', k+q_1 h+s_i, q_0) \right|.
\end{aligned}$$

We bound the  $h = 0$  term using the Weil bound on the individual Kloosterman sums to get

$$H T^2 \ll_{\epsilon, j} K q_1^{j+\epsilon} \left( K q_0^{j+\epsilon} + \sum_{0 < |h| \leq H} \left| \sum_{\substack{k \in J \\ k+q_1 h \in J}} \prod_{i=1}^j S(a', k+s_i, q_0) S(a', k+q_1 h+s_i, q_0) \right| \right).$$

The result follows.  $\square$

The previous lemma bounds  $T$  in terms of sums with twice as many Kloosterman factors. The new shifts are  $s_1, \dots, s_j, s_1+q_1 h, \dots, s_j+q_1 h$ , and the exponential  $e_q(-kM)$ , if it exists, is removed. We will apply it  $l$  times, starting at the sum

$$T = \sum_{k \in J} e_q(-Mk) S(a, k, q).$$

For the remainder of the paper  $T$  will refer to this particular  $j = 1$  case of the above  $T$  whereas  $T(\dots)$  will be one of the more general sums.

**Lemma 4.3.** *Let  $q$  be as in Theorem 1.3 and  $T$  as defined above. We have*

$$\begin{aligned}
T^{2^l} &\ll_{\epsilon, l} q^\epsilon \left( q^{2^{l-1}} \sum_{j=1}^l K^{2^l-2^{l-j}} q_{l-j+1}^{2^{l-j}} \right. \\
&\quad \left. + K^{2^l-l-1} (q/q_0)^{2^{l-1}+1} \sum_{0 < |h_1| \leq K/q_1} \dots \sum_{0 < |h_l| \leq K/q_l} |T(h_1, \dots, h_l)| \right)
\end{aligned}$$

where

$$T(h_1, \dots, h_l) = \sum_{k \in J(h_1, \dots, h_l)} \prod_{I \subseteq \{1, \dots, l\}} S\left(a', k + \sum_{i \in I} q_i h_i, q_0\right),$$

with  $J(h_1, \dots, h_l)$  an interval of length at most  $K$  and  $(a', q) = 1$ .

*Proof.* Observe that by our assumption that  $[q/N] \geq q_i$  we know that  $K \geq q_i$  for  $1 \leq i \leq l$ . This means that the applications of Lemma 4.2 in the following proof are all justified.

We use induction in  $l$ . If  $l = 1$  then applying Lemma 4.2 gives

$$T^2 \ll_{\epsilon} q^{\epsilon} \left( qKq_1 + q_1^2 \sum_{0 < |h_1| \leq K/q_1} \left| \sum_{k \in J(h_1)} S(a', k, q_0) S(a', k + q_1 h_1, q_0) \right| \right),$$

as required.

Now suppose  $l > 1$  and that the result holds for  $l - 1$ . We assume that  $q = q_0 q_1 \dots q_l$  and apply the inductive hypothesis with the factorisation

$$q = r_0 r_1 \dots r_{l-1}$$

where  $r_0 = q_0 q_1$ , and  $r_i = q_{i+1}$  for  $1 \leq i \leq l - 1$ . This results in

$$\begin{aligned} T^{2^{l-1}} \ll_{\epsilon, l} q^{\epsilon} & \left( q^{2^{l-2}} \sum_{j=1}^{l-1} K^{2^{l-1}-2^{l-1-j}} q_{l-j+1}^{2^{l-1-j}} \right. \\ & \left. + K^{2^{l-1}-l} (q/q_0 q_1)^{2^{l-2}+1} \sum_{0 < |h_2| \leq K/q_2} \dots \sum_{0 < |h_l| \leq K/q_l} |T(h_2, \dots, h_l)| \right) \end{aligned}$$

with

$$T(h_2, \dots, h_l) = \sum_{k \in J(h_2, \dots, h_l)} \prod_{I \subseteq \{2, \dots, l\}} S \left( a', k + \sum_{i \in I} q_i h_i, q_0 q_1 \right).$$

Squaring our bound, using Cauchy's inequality on the final sum, we get

$$\begin{aligned} T^{2^l} \ll_{\epsilon, l} q^{\epsilon} & \left( q^{2^{l-1}} \sum_{j=1}^{l-1} K^{2^l-2^{l-j}} q_{l-j+1}^{2^{l-j}} \right. \\ & \left. + K^{2^l-l-1} (q/q_0 q_1)^{2^{l-1}+1} \sum_{0 < |h_2| \leq K/q_2} \dots \sum_{0 < |h_l| \leq K/q_l} |T(h_2, \dots, h_l)|^2 \right). \end{aligned}$$

We now use Lemma 4.2 with  $j = 2^{l-1}$  to get the bound

$$T(h_2, \dots, h_l)^2 \ll_{\epsilon, l} q^{\epsilon} q_1^{2^{l-1}+1} \left( K q_0^{2^{l-1}} + \sum_{0 < |h_1| \leq K/q_1} |T(h_1, \dots, h_l)| \right)$$

where

$$\begin{aligned} T(h_1, \dots, h_l) &= \sum_{k \in J(h_1, \dots, h_l)} \prod_{I \subseteq \{2, \dots, l\}} S\left(a'', k + \sum_{i \in I} q_i h_i, q_0\right) S\left(a'', k + \sum_{i \in I} q_i h_i + q_1 h_1, q_0\right) \\ &= \sum_{k \in J(h_1, \dots, h_l)} \prod_{I \subseteq \{1, \dots, l\}} S\left(a'', k + \sum_{i \in I} q_i h_i, q_0\right), \end{aligned}$$

for some  $(a'', q) = 1$ . Observe that this corresponds precisely to the  $T(h_1, \dots, h_l)$  given in the claim.

We conclude that

$$\begin{aligned} T^{2^l} &\ll_{\epsilon, l} q^\epsilon \left( q^{2^{l-1}} \sum_{j=1}^{l-1} K^{2^l - 2^{l-j}} q_{l-j+1}^{2^{l-j}} + K^{2^l - 1} (q/q_0 q_1)^{2^{l-1}} q_1^{2^{l-1}+1} q_0^{2^{l-1}} \right. \\ &\quad \left. + K^{2^l - l - 1} (q/q_0)^{2^{l-1}+1} \sum_{0 < |h_1| \leq K/q_1} \dots \sum_{0 < |h_l| \leq K/q_l} |T(h_1, \dots, h_l)| \right) \\ &= q^\epsilon \left( q^{2^{l-1}} \sum_{j=1}^l K^{2^l - 2^{l-j}} q_{l-j+1}^{2^{l-j}} \right. \\ &\quad \left. + K^{2^l - l - 1} (q/q_0)^{2^{l-1}+1} \sum_{0 < |h_1| \leq K/q_1} \dots \sum_{0 < |h_l| \leq K/q_l} |T(h_1, \dots, h_l)| \right). \end{aligned}$$

□

### 4.3 Estimating $T(h_1, \dots, h_l)$

It remains to estimate

$$T(h_1, \dots, h_l) = \sum_{k \in J(h_1, \dots, h_l)} \prod_{I \subseteq \{1, \dots, l\}} S\left(a', k + \sum_{i \in I} q_i h_i, q_0\right),$$

where  $(a', q_0) = 1$ .

We begin with the following estimate for complete exponential sums to a prime modulus.

**Lemma 4.4.** *Let  $p$  be a prime,  $(a, p) = 1$  and let  $s_1, \dots, s_j, b$  be integers. We have*

$$\sum_{k \pmod{p}} e_p(-kb) \prod_{i=1}^j S(a, k + s_i, p) \ll_j \begin{cases} p^{\frac{j+2}{2}} & b = 0 \text{ and } E(s_1, \dots, s_j) \\ p^{\frac{j+1}{2}} & \text{otherwise,} \end{cases}$$

where  $E(s_1, \dots, s_j)$  denotes the property that all the  $s_i$  occur with even multiplicity.

*Proof.* The first part follows directly from the Weil bound

$$|S(a, k + s_i, p)| \leq 2\sqrt{p}.$$

For the second part we use a result of Fouvry, Ganguly, Kowalski and Michel [4, Proposition 3.2]. Let

$$\text{Kl}_2(a; p) = \frac{S(a, 1, p)}{p^{1/2}}.$$

If  $k \not\equiv 0 \pmod{p}$  then

$$S(a, k, p) = p^{1/2} \text{Kl}_2(ak; p).$$

We therefore have

$$\sum_{k \pmod{p}} e_p(-kb) \prod_{i=1}^j S(a, k + s_i, p) = p^{j/2} \sum_{\substack{k \pmod{p} \\ k + s_i \not\equiv 0 \pmod{p}}} e_p(-kb) \prod_{i=1}^j \text{Kl}_2(a(k + s_i); p) + O(jp^{j/2}),$$

where the error comes from the terms with  $k + s_i \equiv 0 \pmod{p}$ , for which we can use the Weil bound. The maps  $k \mapsto a(k + s_i)$  are in  $\text{PGL}_2(\mathbb{F}_p)$ . If  $s_i \neq s_j$  then  $k \mapsto a(k + s_i), k \mapsto a(k + s_j)$  are different.

If  $b = 0$  then [4, Proposition 3.2] states that if  $\beta_1, \dots, \beta_j \in \text{PGL}_2(\mathbb{F}_p)$  then, provided the multiplicities of the  $\beta_i$  are not all even, we have

$$\sum_{\substack{k \pmod{p} \\ \beta_i k \neq 0, \infty}} e_p(-kb) \prod_{i=1}^j \text{Kl}_2(\beta_i k; p) \ll_j p^{1/2}.$$

If  $b \neq 0$  then the same bound can be shown to hold for all choices of  $\beta_i$ . The proof involves some small modifications to the argument from [4], detailed by Kowalski in a private communication.  $\square$

We use this in conjunction with the following combinatorial result.

**Lemma 4.5.** *Let  $p \geq 3$  be prime and let  $h_1, \dots, h_l \in \mathbb{F}_p$ . Suppose that the  $2^l$  sums*

$$\sum_{i \in I} h_i \text{ for } I \subseteq \{1, \dots, l\}$$

*form a list of elements all of whose entries have even multiplicities. At least one of the  $h_i$  must then be 0.*

*Proof.* Let  $\omega = e_p(1)$  and consider the algebraic integer  $\alpha \in \mathbb{Z}[\omega]$  given by

$$\alpha = \prod_{i=1}^l (1 + \omega^{h_i}) = \sum_{I \subseteq \{1, \dots, l\}} \omega^{\sum_{i \in I} h_i}.$$



Our assumption that the  $\sum_{i \in I} h_i$  all have even multiplicities therefore implies that  $\alpha$  is a sum of even multiples of powers of  $\omega$ . In particular  $2|N_{\mathbb{Q}(\omega)/\mathbb{Q}}(\alpha)$ .

If  $h_i \not\equiv 0 \pmod{p}$  then since  $p \geq 3$  it is well known that

$$N_{\mathbb{Q}(\omega)/\mathbb{Q}}(1 + \omega^{h_i}) = 1.$$

We therefore have a contradiction unless at least one of the  $h_i$  is 0.  $\square$

Combining the last two lemmas we immediately deduce the following.

**Lemma 4.6.** *Let  $p$  be prime,  $(a, p) = 1$  and let  $h_1, \dots, h_l, b$  be integers. We have*

$$\sum_{k \pmod{p}} e_p(-kb) \prod_{I \subseteq \{1, \dots, l\}} S\left(a, k + \sum_{i \in I} h_i, p\right) \ll_l p^{\frac{2^l+1}{2}}(p, b, \prod h_i)^{1/2}.$$

Next we generalise this to squarefree moduli.

**Lemma 4.7.** *Let  $q$  be squarefree,  $(a, q) = 1$  and let  $h_1, \dots, h_l, b$  be integers. For any  $\epsilon > 0$  we have*

$$\sum_{k \pmod{q}} e_q(-kb) \prod_{I \subseteq \{1, \dots, l\}} S\left(a, k + \sum_{i \in I} h_i, q\right) \ll_{\epsilon, l} q^{\frac{2^l+1}{2} + \epsilon}(q, b, \prod h_i)^{1/2}.$$

*Proof.* The sum has a multiplicative property. Specifically, if  $(q_0, q_1) = 1$  then

$$\begin{aligned} & \sum_{k \pmod{q_0 q_1}} e_{q_0 q_1}(-kb) \prod_{I \subseteq \{1, \dots, l\}} S\left(a, k + \sum_{i \in I} h_i, q_0 q_1\right) \\ &= \sum_{\substack{k_0 \pmod{q_0} \\ k_1 \pmod{q_1}}} e_{q_0}(-bk_0 \overline{q_1}) e_{q_1}(-bk_1 \overline{q_0}) \prod_{I \subseteq \{1, \dots, l\}} S(a, k_0 q_1 \overline{q_1} + k_1 q_0 \overline{q_0} + \sum_{i \in I} h_i, q_0 q_1) \\ &= \sum_{k_0 \pmod{q_0}} e_{q_0}(-bk_0 \overline{q_1}) \prod_{I \subseteq \{1, \dots, l\}} S\left(a \overline{q_1}, (k_0 q_1 \overline{q_1} + \sum_{i \in I} h_i) \overline{q_1}, q_0\right) \\ & \quad \times \sum_{k_1 \pmod{q_1}} e_{q_1}(-bk_1 \overline{q_0}) \prod_{I \subseteq \{1, \dots, l\}} S\left(a \overline{q_0}, (k_1 q_0 \overline{q_0} + \sum_{i \in I} h_i) \overline{q_0}, q_1\right). \end{aligned}$$

It follows that if  $q$  is squarefree we may factorise the sum as a product over  $p|q$  of sums to modulus  $p$ . Each sum may then be estimated using the last lemma. The integers  $b, h_i$  occurring in the factors are different to those in our sum to modulus  $q$ , however the changes are simply by multiplicative factors coprime to  $q$ . It follows that each factor may be bounded by

$$C_l p^{\frac{2^l+1}{2}}(p, b, \prod h_i)^{1/2}$$

for some constant  $C_l$ , whence our sum is bounded by

$$\prod_{p|q} \left( C_l p^{\frac{2^l+1}{2}}(p, b, \prod h_i)^{1/2} \right) \ll_{\epsilon, l} q^{\frac{2^l+1}{2}+\epsilon} (q, b, \prod h_i)^{1/2}.$$

□

We now return to our sum

$$\begin{aligned} T(h_1, \dots, h_l) &= \sum_{k \in J(h_1, \dots, h_l)} \prod_{I \subseteq \{1, \dots, l\}} S \left( a', k + \sum_{i \in I} q_i h_i, q_0 \right) \\ &= \sum_{j \pmod{q_0}} \sum_{\substack{k \in J(h_1, \dots, h_l) \\ k \equiv j \pmod{q_0}}} \prod_{I \subseteq \{1, \dots, l\}} S \left( a', j + \sum_{i \in I} q_i h_i, q_0 \right) \\ &= \frac{1}{q_0} \sum_{b \pmod{q_0}} h(b) \sum_{j \pmod{q_0}} e_{q_0}(bj) \prod_{I \subseteq \{1, \dots, l\}} S \left( a', j + \sum_{i \in I} q_i h_i, q_0 \right), \end{aligned}$$

with

$$h(b) = \sum_{k \in J(h_1, \dots, h_l)} e_{q_0}(-bk).$$

We have the standard estimate

$$h(b) \ll \min \left( K, \frac{1}{\|b/q_0\|} \right).$$

Since  $(q_i, q_0) = 1$  for  $i \neq 0$  we can write

$$(q_0, b, \prod q_i h_i) = (q_0, b, \prod h_i).$$

We may therefore use the last lemma to obtain

$$T(h_1, \dots, h_l) \ll_{\epsilon, l} q_0^{\frac{2^l+1}{2}+\epsilon} \cdot \frac{1}{q_0} \sum_{b \pmod{q_0}} \min \left( K, \frac{1}{\|b/q_0\|} \right) (q_0, b, \prod h_i)^{1/2}.$$

Finally we estimate

$$\begin{aligned}
& \frac{1}{q_0} \sum_{b \pmod{q_0}} \min \left( K, \frac{1}{\|b/q_0\|} \right) (q_0, b, \prod h_i)^{1/2} \\
& \ll \frac{K}{q_0} (q_0, \prod h_i)^{1/2} + \sum_{0 < b \leq q_0/2} \frac{1}{b} (q_0, b, \prod h_i)^{1/2} \\
& = \frac{K}{q_0} (q_0, \prod h_i)^{1/2} + \sum_{d|(q_0, \prod h_i)} d^{1/2} \sum_{\substack{0 < b \leq q_0/2 \\ (q_0, b, \prod h_i) = d}} \frac{1}{b} \\
& \ll_{\epsilon} \frac{K}{q_0} (q_0, \prod h_i)^{1/2} + q^{\epsilon} \\
& \ll_{\epsilon} q^{\epsilon} (q_0, \prod h_i)^{1/2} \left( \frac{K}{q_0} + 1 \right)
\end{aligned}$$

so we conclude that

$$T(h_1, \dots, h_l) \ll_{\epsilon, l} q^{\epsilon} \left( \frac{K}{q_0} + 1 \right) q_0^{\frac{2^l+1}{2}} (q_0, \prod h_i)^{1/2}.$$

#### 4.4 Conclusion

Inserting the above bound for  $T(h_1, \dots, h_l)$  into the result of Lemma 4.3 we obtain

$$\begin{aligned}
T^{2^l} & \ll_{\epsilon, l} q^{\epsilon} \left( q^{2^{l-1}} \sum_{j=1}^l K^{2^l - 2^{l-j}} q_{l-j+1}^{2^{l-j}} \right. \\
& \quad \left. + \left( \frac{K}{q_0} + 1 \right) K^{2^l - l - 1} (q/q_0)^{2^{l-1}+1} q_0^{\frac{2^l+1}{2}} \sum_{0 < |h_1| \leq K/q_1} \dots \sum_{0 < |h_l| \leq K/q_l} (q_0, \prod h_i)^{1/2} \right).
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{0 < |h_1| \leq K/q_1} \dots \sum_{0 < |h_l| \leq K/q_l} (q_0, \prod h_i)^{1/2} & \leq \sum_{0 < |h| \leq K^l/(q_1 \dots q_l)} \tau_l(h) (q_0, h)^{1/2} \\
& \ll_{\epsilon, l} q^{\epsilon} \sum_{0 < |h| \leq K^l/(q_1 \dots q_l)} (q_0, h)^{1/2} \\
& \ll_{\epsilon} q^{\epsilon} K^l / (q_1 \dots q_l) \\
& = q^{-1+\epsilon} K^l q_0.
\end{aligned}$$

We conclude that

$$T^{2^l} \ll_{\epsilon, l} q^{2^{l-1}+\epsilon} \left( \sum_{j=1}^l K^{2^l-2^{l-j}} q_{l-j+1}^{2^{l-j}} + \left( \frac{K}{q_0} + 1 \right) K^{2^l-1} q_0^{1/2} \right).$$

Lemma 4.1 now follows, on recalling that the sums  $S(r)$  were a special case of the sum  $T$ , and therefore Theorem 1.3 is proven.

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